2009 Chicago Area All-Star Math Team Tryouts
Solutions

1. If a car sells for $q \cdot \$1000$ and the salesman earns $q\% = q/100$, he earns $\$10q^2$. He earns an additional \$100 per car, and he sells p cars, so his total earnings are $p(\$100 + \$10q^2)$. This totals \$4600. Now divide both sides by \$10 to obtain the equation $p(10 + q^2) = 460$. Since p and q are positive integers, we know that p and $10 + q^2$ are factors of 460. The factors of 460 are 1, 2, 4, 5, 10, 20, 23, 46, 92, 115, 230, and 460. Since q is a positive integer the only of those factors that can be $10 + q^2$ is $46 = 10 + 6^2$. So $q = 6$ and $p = 10$, and the ordered pair is $\boxed{(10, 6)}$.

2. The slope between $(-2, x^2 + 2x)$ and $(x, 4x + 8)$ is $\dfrac{(x^2+2x)-(4x+8)}{-2-x} = -\dfrac{x^2-2x-8}{2+x}$. Now $x^2 - 2x - 8$ factors as $(x - 4)(x + 2)$, so the fraction simplifies to $-(x - 4)$, $x \neq -2$. Setting this equal to $-3/4$ as given, we get $x - 4 = 3/4$, or $x = \boxed{19/4}$.

3. Since we have to take the square root of it, let's see if we can show that $36+10\sqrt{11}$ is actually a nice square. Let's try to find a, b, and c so that $(a+b\sqrt{c})^2 = 36+10\sqrt{11}$. Now $(a+b\sqrt{c})^2 = (a^2+b^2c)+2ab\sqrt{c}$. So right away, we know $c = 11$. We also know that $2ab$ equals 10, so $ab = 5$. Since b can't be 5 (otherwise b^2c is way bigger than 36), we try $a = 5$ and $b = 1$, and it works. So $36+10\sqrt{11} = (5+\sqrt{11})^2$ and similarly $36-10\sqrt{11} = (5-\sqrt{11})^2$. Putting these into the original problem gives:

$$(5+\sqrt{11})^3-(5-\sqrt{11})^3 = (125+75\sqrt{11}+165+11\sqrt{11})-(125-75\sqrt{11}+165-11\sqrt{11}) = \boxed{172\sqrt{11}}.$$

(You can avoid cubing these things if you factor the difference of cubes.)

4. Using the laws of logarithms, $\log_2 x^2 = 2\log_2 x$ and $\log_x 4 = \dfrac{\log_2 4}{\log_2 x} = \dfrac{2}{\log_2 x}$. If we let $z = \log_2 x$, our expression simplifies to $2z + \dfrac{2}{z}$. By the AM-GM inequality, we find that $\dfrac{z+1/z}{2} \geq \sqrt{z\dfrac{1}{z}} = 1$ and the only way to get equality is if $z = 1/z$. So we see that $z = 1$ (which means $x = 2$, which is bigger than one, so we meet the other stated requirement). Then the final value is $2z + 2/z = \boxed{4}$.

5. Note that $MB = 6$, and the altitude to point Q in ΔQMB is 20, while the altitude to Q in ΔDQC must be 10. This makes the height of the parallelogram perpendicular to bases $\overline{AB}$ and $\overline{CD}$ 30, so the parallelogram has area $12 \cdot 30 = 360$. Additionally, ΔAMD has base 6 and height 30, so area 90. Thus the area we seek is $360 - 90 - 60 - 60 = \boxed{150}$.

6. There is a standard approach that often works for messy combinations of trig functions like this. Look to write the arguments of the trig functions as sums and differences of common numbers. In this case, rewrite $\frac{\sin x - \sin 7x}{\cos 7x - \cos x} = \frac{\sin(4x-3x) - \sin(4x+3x)}{\cos(4x+3x) - \cos(4x-3x)}$. Now you use the angle addition formulas to get $\frac{\sin 4x \cos 3x - \cos 4x \sin 3x - \sin 4x \cos 3x - \cos 4x \sin 3x}{\cos 4x \cos 3x - \sin 4x \sin 3x - \cos 4x \cos 3x - \sin 4x \sin 3x}$.
Next, collect like terms on top and bottom: $\frac{-2\cos 4x \sin 3x}{-2\sin 4x \sin 3x}$ and simplify to get cot 4x. So we really need to solve cot 4x = tan 6x. But cot a = tan(π/2 – a), so we need tan(π/2 – 4x) = tan 6x. So, taking into account the periodicity of tangent, we need the smallest positive x that solves the equation π/2 – 4x + kπ = 6x, or 10x = π/2 + kπ. We make the right hand side positive and as small as possible by using k = 0, and thus find that x = $\boxed{\pi/20}$.

7. We want to find a number a so that $f(a)$ = 27. Now if 5x – 2 = 27, then x = 29/5. So now we know that $f\left(\frac{2(29/5)-3}{(29/5)-2}\right) = 27$. Simplifying by multiplying top and bottom by five, we find f(43/19) = 27, so a = $\boxed{43/19}$.

8. Let P be the center of the larger circle, and Q the center of the smaller circle. Sketch in segments $\overline{FP}$ and $\overline{CQ}$. We are given that $m\angle FPB$ = 82°. By similarity, $m\angle CQB$ is also 82°. Because of the tangency condition, $m\angle QCB$ = 90°. Thus $m\angle CBQ$ = 8° and this is also the measure of $\angle ABD$. Now an arc seen from a point on the circle is always subtends half the angle it would if it were seen from the center of the circle, so $m\angle ABD$ = ½ $m\angle APD$, and thus the measure of arc AD is $\boxed{16°}$.

9. If we can determine how many combinations of assignments of teams to lanes there are, we simply need to multiply by 36 to find the total number of arrangements of swimmers. This is because each team's three swimmers can be arranged in six ways (ABC, ACB, BAC, BCA, CAB, CBA). So let's just brute-force count the lane arrangements. The notations below show which lanes team 1 is in, together with which lanes are possible to be empty; team 2 fills the remaining lanes. Keep in mind that no two adjacent lanes can belong to the same team. So the lanes that could be occupied by team 1 are: 135(6 or 7 empty), 136(4 or 5 empty), 137(5), 146(2 or 3), 157(3), 246(1, 3, 5, or 7), 247(5 or 6), 257(3 or 4), 357(1 or 2), for a total of 18 arrangments. Thus the total number of arrangements of swimmers is 18·36 = $\boxed{648}$.

10. If the series is $21 + 21r + 21r^2 + \cdots$ then the sum is $\frac{21}{1-r} = \frac{21^2}{21-21r}$. Since this is an integer and 21r is a positive integer, we need 21 – 21r to divide evenly into 21^2. Also, 0 < r < 1, so 21r < 21. These together mean that 21 – 21r must be 1, 3, 7, or 9, making the second term 21r = $\boxed{12, 14, 18, \text{or } 20}$.

11. One way to go about this is to denote $m\angle PDM = m\angle PDL$ by x and $m\angle PEM = m\angle PEM$ by y. Then we start chasing through the diagram using the facts that supplementary angles add to 180°, as do the interior angles of a triangle. Let S be the intersection of $\overline{EP}$ and $\overline{DM}$, and T the intersection of $\overline{EM}$ and $\overline{DP}$. So $m\angle RSE = 66 - y$, and $m\angle MSE = 114 + y$. Next, $m\angle RME = 66 - 2y$, so $m\angle MTD = 114 + 2y - x$. Now the four angles in quadrilateral $PSMT$ add to 360, so $72 + 114 + y + 66 - 2y + 114 + 2y - x = 360$, or $x - y = 6$. Now $m\angle LTD = 66 - 2y + x$, so in ΔDLM we add the angles to get $m\angle DLM + x + 66 - 2y + x = 180$, or $m\angle ALM = m\angle DLM = 114 + 2y - 2x = \boxed{102°}$.

Note: using similar logic, if specific angle measures are not given, you can prove the surprising invariant $m\angle LAR + m\angle LMR = 2\, m\angle EPD$.

12. First, let $y = \sin x + \cos x$ and $z = \cos x - \sin x$. Now we need $\sin(y) = \cos(z)$. Let's use the relation between sine and cosine to change this to $\sin(y) = \sin(\pi/2 + z)$. Thus either $y = \pi/2 + z + 2k\pi$ or $y = \pi - (\pi/2 + z + 2k\pi)$. The first simplifies to $y - z = \pi/2 + 2k\pi$, and the second simplifies to $y + z = \pi/2 - 2k\pi$.

Now $y - z = 2\sin x$, which ranges from -2 to 2. But $-3\pi/2$ is less than -2 while $5\pi/2$ is larger than 2, so in the first possibility we must have $k = 0$, and $2\sin x = \pi/2$, so $\sin x = \pi/4$.

In the second case, $y + z = 2\cos x$, which also ranges from -2 to 2. By similar reasoning, we must again have $k = 0$, so $\cos x = \pi/4$. If $\cos x = \pi/4 > 3/4$, then $\sin x = \pm\sqrt{1-\cos^2 x} < \sqrt{7}/4$. Since $\sqrt{7} < \pi$, this is clearly smaller than the answer in the previous case, and we are looking for the largest possible answer.

So our final conclusion is that the largest possible value of $\sin x = \boxed{\pi/4}$.

13. There are a total of eight lines, so there were originally $8 - k$ lines drawn. How many triangles are made by these lines alone? Well, because any two of these lines intersect (none are parallel) and no three are concurrent, any three of these lines determine a unique triangle. So these lines themselves determine $\binom{8-k}{3}$ triangles.

Now no triangle has two sides among the k parallel lines, but we can take any one of these lines together with two of the original lines to determine a triangle. This accounts for $k\binom{8-k}{2}$ more triangles. So we need $\binom{8-k}{3} + k\binom{8-k}{2} = 40$.

We could multiply out the combinations, but it's probably easier just to try various numbers. If $k = 0$, we'd get 56 triangles; $k = 1$ gives 56 triangles; $k = 2$ leads to 50 triangles. If we try $k = 3$ we get 40 triangles while $k = 4$ leads to 28 triangles. Larger k's gives smaller numbers of triangles, so we hit it when we found $k = \boxed{3}$.

14. Basically, we have to count the number of different factors of numbers which have the form $42 - 12x$.. So let's do it. If $x = 1$, we need to solve $by = 30$ which has positive integer solutions as long as b is among 1, 2, 3, 5, 6, 10, 15, and 30. If $x = 2$, we solve $by = 18$, so b must be among 1, 2, 3, 6, 9, 18. If $x = 3$, then $by = 6$ so b is one of 1, 2, 3, 6. Larger x's make $42 - 12x$ negative, so b and y can't both be positive. So the collection of b's that work is 1, 2, 3, 5, 6, 9, 10, 15, 18, and 30, a collection of $\boxed{10}$ numbers.

15. Let n be factored into primes as $p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. Then n has $(a_1 + 1)(a_2 + 1)\cdots(a_r + 1)$ factors. Also, $n^2 = p_1^{2a_1} p_2^{2a_2} \cdots p_r^{2a_r}$ so n^2 has $(2a_1 + 1)(2a_2 + 1)\cdots(2a_r + 1)$ factors. So we know $(2a_1 + 1)(2a_2 + 1)\cdots(2a_r + 1) = 2009 = 7\cdot 7\cdot 41$. This leads to the following possibilities: $a_1 = 1004$, and n has 1005 factors; $a_1 = 3$ and $a_2 = 143$ giving n 576 factors; $a_1 = 24$ and $a_2 = 20$ giving n 525 factors; $a_1 = 3$, $a_2 = 3$, and $a_3 = 20$ giving n the smallest possibility, $\boxed{336}$ factors.

16. First, there are $\binom{17}{5} = \frac{17!}{12!5!}$ ways to choose five numbers. Now from 1 to 17 there are six number which can be written as $3a + 1$ (namely 1, 4, 7, 10, 13, and 16), six which can be written as $3a - 1$ (2, 5, 8, 11, 14, and 17), and five which are divisible by 3. Let's call the first group +1's, the second -1's, and the third 0's. For the sum of five numbers to be divisible by three, you must have: all five 0's; three 0's, a +1, and a -1; one 0 and two each of +1 and -1; three +1's and two 0's; four +1's and a -1; three -1's and two 0's; or four -1's and a +1.

Thus we have the following total of ways to get a sum divisible by three:

$$\binom{5}{5}\binom{6}{0}\binom{6}{0}+\binom{5}{3}\binom{6}{1}\binom{6}{1}+\binom{5}{1}\binom{6}{2}\binom{6}{2}+\binom{5}{2}\binom{6}{3}\binom{6}{0}+\binom{5}{0}\binom{6}{4}\binom{6}{1}+\binom{5}{2}\binom{6}{0}\binom{6}{3}+\binom{5}{0}\binom{6}{1}\binom{6}{4}$$

This comes to 1 + 360 + 1125 + 200 + 90 + 200 + 90 = 2066 different choices that have a sum divisible by three. So our final answer is $\frac{2066}{17!/12!5!} = \frac{2066\cdot 12!5!}{17!} = \frac{2066\cdot 5!}{13\cdot 14\cdot 15\cdot 16\cdot 17} = \boxed{\frac{1033}{3094}}$.

17. Quick! Set $x = -1$. Then $x + 2 = 1$, so the right-hand side is $a + b + c + d$, and the left-hand side evaluates to $\boxed{28}$.

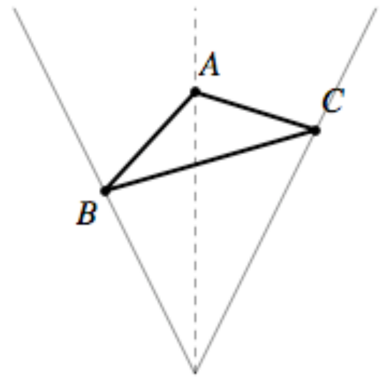
18. Let's look at the down clues. 1 down can be 128, 256, or 512 (and 6 across instantly rules out 512 because the digits cannot decrease from 2!). 2 down can only be 287, and now since the digits in 6 across must decrease, 256 is ruled out for 1 down. So we already know the left-most six digits, as shown.

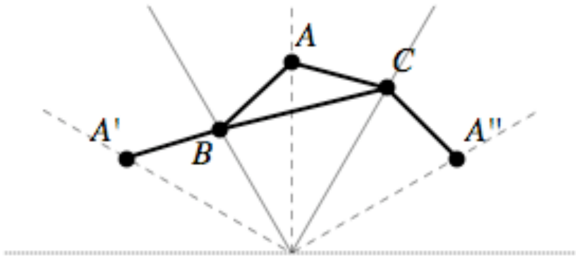
1	2		
2	8		
8	7		

Now 1 across must be 1211 or 1222. The three-digit squares starting with 1 or 2 are 100, 121, 144, 169, 196, 225, 256, and 289. We can rule out 169, 196, 256, and 289 for 4 down because then the digits in 6 across couldn't decrease. We can also rule out 121 and 225 because otherwise the digits of 5 across wouldn't be all different. This leaves 100 and 144. So 1 across is 1211. Now the doubles of squares with first digit 1 are 128 and 162. Again, we rule out 128 because of 5 across. So 3 down is 162. Then we rule out 144 for 4 down because of the decreasing digits in 6 across. Thus the completed puzzle is as shown.

1	2	1	1
2	8	6	0
8	7	2	0

19. The goal of this problem is to locate points B and C as shown in the diagram so as to minimize the perimeter of the triangle. While there are geometric techniques and calculus techniques to do this, a very much nicer approach is to use mirror images, as shown below, where the original picture has been reflected over both rays of the original angle.. Now the perimeter of the triangle is the same as the disance from A' to B to C to A''.





Where can we put B and C to minimize this distance? Since a straight line is the shortest distance between two points, we shoud draw the straight line from A' to A''! Then place B and C where this line meets the sides of the original angle. Now, since the perimeter of the triangle the same as the distance from A' to B to C to A'', and this is now a straight line, we just have to figure out this distance. But $A'A''$ is the length of the base of an isosceles triangle whose vertex angle is 120° and side length 12. So we use the law of cosines to get $A'A''^2 = 12^2 + 12^2 - 2\cdot 12\cdot 12\cdot\cos(120°) = 432$, so the minimum perimeter is $\boxed{12\sqrt{3}}$.

20. The distance from z to z^2 in the complex plane is $|z - z^1|$. So we know $|z - z^2| = |1 - z|/2$. So dividing, we get $\dfrac{|z-z^2|}{|1-z|} = \dfrac{1}{2}$. But the left-hand side simplifies:

$\dfrac{|z-z^2|}{|1-z|} = \left|\dfrac{z-z^2}{1-z}\right| = |z|$. So $|z| = ½$. We also know that $|1 - z| = |1 - z^2|$, so dividing gives

$\dfrac{|1-z^2|}{|1-z|} = \left|\dfrac{1-z^2}{1-z}\right| = |1+z|$. So we get $|1 + z| = 1$. So now we know that $a^2 + b^2 = (½)^2 = ¼$ and $(1 + a)^2 + b^2 = 1$. So we subtract to find that $1 + 2a = ¾$, so $a = -1/8$. Thus $b^2 = 1/4 - 1/64$. So $b^2 = 15/64$, and $b > 0$, so $b = \boxed{\sqrt{15}/8}$.

21. For a system of three linear equations in three variables to not have a unique solutions, the determinant of the coefficients must be zero. So we compute:

$$\begin{vmatrix} 4 & 3 & 0 \\ 2 & 0 & 5 \\ 0 & 3 & -a \end{vmatrix} = 6a - 60 = 0.$$

This means $a = 10$. Now the system is *singular* and either has zero or infinitely many solutions depending on the right-hand side. On the left, the third equation plus twice the second equals the first, so to have solutions, the right hand side must be the same. Thus we must have $4 + 2k = 15$, so $k = \boxed{11/2}$.

22. Here's a sneaky trick: rewrite the equation as $4a^2 - (b^2 - b) = 30$. Now complete the square: $4a^2 - (b^2 - b + \frac{1}{4}) = 29\frac{3}{4}$. Multiply by four to get rid of the fractions, and we end up with $16a^2 - (2b - 1)^2 = 119$.

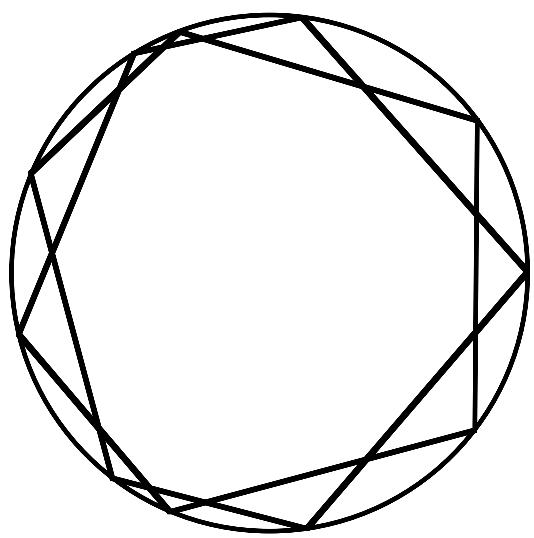
Now a difference of squares can be easily factored: $x^2 - y^2 = (x + y)(x - y)$. So we factor our equation to obtain $(4a - 2b + 1)(4a + 2b - 1) = 119$. Now everything on the left is an integer, so we factor 119 into integers to obtain $119 = 7\cdot17$. So we make all possible combinations: 119 can equal $1\cdot119$, $7\cdot17$, $-1\cdot-119$, etc. Here are the possibilities:

$4a - 2b + 1$	$4a + 2b - 1$	a	b
1	119	15	30
7	17	3	3
17	7	3	-2
119	1	15	-29
-1	-119	-15	-30
-7	-17	-3	3
-17	-7	-3	2
-119	-1	-15	29

Since all solutions are integers, we see that there are **8** ordered pairs (a, b) that will work.

23. Not all the points have the same x-coordinate, so the line will not be vertical. Thus the line has a slope, so we simply need to set the slopes between the pairs of points equal. Thus $\frac{b-8}{b-4} = \frac{2b-16}{b+4}$. Cross-multiplying, we find that $b^2 - 4b - 32 = 2b^2 - 24b + 64$, or $B^2 - 20b + 96 = 0$. This factors as $(b - 12)(b - 8) = 0$, so **$b = 12$ or $b = 8$**.

24. Let's say you start at A_1 and walk around the figure (shown is one example, other spacings of the points are possible). Now when you reach A_3, you turn toward A_5 by turning through the supplement of angle A_3. That is, you turn through an angle of $180° - m\angle A_3$. Keep going around the figure and you will turn through angles of $180° - m\angle A_5$, $180° - m\angle A_7$, etc. When you return to A_1, you should turn through $180° - m\angle A_1$ so that you are facing your original direction.



But you have turned around two full times, so that we have $11\cdot180° - m\angle A_1 - m\angle A_3 - m\angle A_3 \cdots - \angle A_{11} = 720°$. This means that the sum of the angles is always $11\cdot180° - 720° =$ **1260°** no matter how the points are spaced around the circle. (Also acceptable is the radian equivalent, $7\pi/2$.)

Alternately, each of the angles cuts off seven of the arc segments that join the points. For instance, $\angle A_1$ cuts off the arcs $\overset{\frown}{A_3A_4}, \overset{\frown}{A_4A_5}, \ldots, \overset{\frown}{A_9A_{10}}$. Each segment of arc is cut off by seven angles. So if we add all the angles, we get each piece of arc seven times, totaling $7\cdot360° = 2520°$. Since each angle measures ½ of the total arc it cuts off, we get that the sum of the angles is ½ or 2520°, or 1260°.